ON THE BOUNDARY OF MODULI SPACES OF LOG HODGE STRUCTURES, II: NONTRIVIAL TORSORS

TATSUKI HAYAMA

Abstract. This is a continuous work of our previous paper. In the previous work we showed a triviality of the torsors in the case where period domains are Hermitian symmetric and a non-triviality for one-example. In this paper we determine whether the torsors are trivial or not for any period domains for pure Hodge structures. We also show a generalization of a previous result which gives a non-triviality on some open sets connecting to cycle spaces.

1. Introduction

Let D be a period domain defined by Griffiths [G]. A variation of $\mathbb{Z}$-Hodge structure over the n-product of punctured disk $(\Delta^*)^n$ gives the period map $(\Delta^*)^n \to \Gamma\backslash D$ where Γ is the monodromy group, i.e. the $\mathbb{Z}$-module generated by monodromy transformations. We assume the monodromy transformations are unipotent. This paper treat partial compactifications of $\Gamma\backslash D$ so that the period map is extended over Δ^n.

Classically, in the case where D is Hermitian symmetric, Ash, Mumford, Rapoport and Tai [AMRT] give partial compactifications of $\Gamma\backslash D$ (and also give compactifications of arithmetic quotient of D). Later, Kato and Usui [KU] generalize toroidal partial compactifications for any period domain D, which is not Hermitian symmetric in general, and show these are moduli spaces of log Hodge structures (Recently Kato, Nakayama and Usui [KNU] generalize it for the setting of mixed Hodge structures). This partial compactification is given by using toroidal embedding associated to the cone generated by the date of the monodromy. In fact, for a generator $T_1, \dots, T_n$ of the monodromy group Γ, the partial compactification $\Gamma\backslash D_\sigma$ is given by the cone $\sigma = \sum_{j=1}^n \mathbb{R}_{\geq 0} N_j$ ($N_j = \log T_j$) in the Lie algebra. Here boundary points are nilpotent orbits associated to a face of σ (see §3.1).

In the "classical" situation, $\Gamma\backslash D_\sigma$ is an analytic space. However $\Gamma\backslash D_\sigma$ can not be an analytic space in the "non-classical" situation. There can be *slits* on the boundary of $\Gamma\backslash D_\sigma$. In fact the codimension of a boundary of $\Gamma\backslash D_\sigma$ can be higher than 1 in the non-classical situation although it is 1 in the classical situation (see Example 3.3). [KU] defines logarithmic manifolds as a generalization of analytic spaces and state $\Gamma\backslash D_\sigma$ is a logarithmic manifold.

1.1. **Main result.** Our main result is about a geometric property of $\Gamma\backslash D_\sigma$. The geometric structure of $\Gamma\backslash D_\sigma$ is given by the map $E_\sigma \to \Gamma\backslash D_\sigma$ (see §3.1). By [KU], $E_\sigma \to \Gamma\backslash D_\sigma$ is a torsor in the category of log manifolds. In the previous paper [H], we showed that this torsor is trivial if D is Hermitian symmetric [H, Theorem 5.6] and moreover the torsor is not trivial for one-example in the non

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classical situation [H, Proposition 5.8]. Generalizing this previous result, we show the following theorem:

Theorem 1.1 (Theorem 3.4)**.** *$E_\sigma \to \Gamma\backslash D_\sigma$ is trivial if and only if D is Hermitian symmetric or $\sigma = \{0\}$.*

This proof is given by that any holomorphic function on D is constant if D is not Hermitian symmetric, which is obtained by applying a result of the book of Fels, Huckleberry and Wolf [FHW] to period domains. From this result, we have $H^0(\Gamma\backslash D_\sigma, \mathcal{O}(E_\sigma))$ vanishes if D is not Hermitian symmetric and $\sigma \neq \{0\}$. The higher degree of the cohomology group $H^p(\Gamma\backslash D_\sigma, \mathcal{O}(E_\sigma))$ is still a problem.

1.2. **Remarks on** [H]**.** In [H, Proposition 5.8], we give a non-triviality of the torsor by a different way from this main result. We show a generalization of it in Proposition 4.3 which gives a non-triviality on some open sets. This statement is written by using the cycle space and the SL(2)-orbit arising from a nilpotent orbit. There a property of the cycle space (Lemma 4.1) induces Proposition 4.3.

As in Proposition 4.3, we expect that properties of cycle spaces have importance on the study of moduli space of log Hodge structures. In §4.3 we observe a property of cycle spaces connecting to Lemma 4.1 for the case of weight 3 with Hodge numbers $h^{3,0} = h^{0,3} = 1$, $h^{2,1} = h^{1,2} = 1$, 0 otherwise. This case is non-classical and types of degenerations are totally classified (Example 3.3). Geometrically this case is corresponding to the quintic-mirror family or Borcea-Voisin mirror family (see [GGK, Part III. A], [U]).

. In this paper, we review period domains in §2.1 and moduli spaces of log Hodge structures in §3.1. We show the main result in §3.2 more precisely. Moreover we review cycle spaces in §2.2 –§2.3, and show the results related to the remarks on [H] in §4.

2. Cycle spaces of period domains

2.1. **Polarized Hodge structures and period domains.** We recall the definition of polarized Hodge structures and of period domains. A Hodge structure of weight w with Hodge numbers $(h^{p,q})_{p,q}$ is a pair $(H_\mathbb{Z}, F)$ consisting of a free $\mathbb{Z}$-module of rank $\sum_{p,q} h^{p,q}$ and of a decreasing filtration on $H_\mathbb{C} := H_\mathbb{Z} \otimes \mathbb{C}$ satisfying the following conditions:

(H1) $\dim_\mathbb{C} F^p = \sum_{r \geq p} h^{r,w-r}$ for all p;
(H2) $H_\mathbb{C} = \bigoplus_{p+q=w} H^{p,q} \quad (H^{p,q} := F^p \cap \overline{F^{w-p}})$.

A polarization $\langle\ ,\ \rangle$ for a Hodge structure $(H_\mathbb{Z}, F)$ of weight w is a non-degenerate bilinear form on $H_\mathbb{Q} := H \otimes \mathbb{Q}$, symmetric if w is even and skew-symmetric if w is odd, satisfying the following conditions:

(P1) $\langle F^p, F^q \rangle = 0$ for $p + q > w$;
(P2) $i^{p-q}\langle v, \bar{v} \rangle > 0$ for $v \in H^{p,q}$.

We fix a polarized Hodge structure $(H_{\mathbb{Z},0}, F_0, \langle\ ,\ \rangle_0)$ of weight w with Hodge numbers $(h^{p,q})_{p,q}$. We define the set of all Hodge structures of this type

$$D := \left\{ F \;\middle|\; \begin{array}{r} (H_{\mathbb{Z},0}, F, \langle\ ,\ \rangle_0) \text{ is a polarized Hodge structure} \\ \text{of weight } w \text{ with Hodge numbers } (h^{p,q})_{p,q} \end{array} \right\}.$$

D is called a period domain. Moreover, we have the flag manifold

$$\check{D} := \left\{ F \;\middle|\; \begin{array}{r} (H_{\mathbb{Z},0}, F, \langle\ ,\ \rangle_0) \text{ satisfies the conditions} \\ \text{(H1), (H2) and (P1)} \end{array} \right\}.$$

$\check{D}$ is called the compact dual of D. D is contained in $\check{D}$ as an open subset. D and $\check{D}$ are homogeneous spaces under the natural actions of $G_{\mathbb{R}}$ and $G_{\mathbb{C}}$ respectively, where $G_A := \mathrm{Aut}\,(H_{A,0}, \langle\ ,\ \rangle_0)$. $G_{\mathbb{R}}$ is a classical group such that

$$G_{\mathbb{R}} \cong \begin{cases} Sp(h,\mathbb{R}) & \text{if } w \text{ is odd,} \\ SO(h_{\mathrm{odd}}, h_{\mathrm{even}}) & \text{if } w \text{ is even,} \end{cases}$$

where $2h = \mathrm{rank}\, H_{\mathbb{Z}}$, $Sp(h,\mathbb{R})$ is the $(2h \times 2h)$-matrix symplectic group, $h_{\mathrm{odd}} = \sum_{\mathrm{p:odd}} h^{p,q}$ and $h_{\mathrm{even}} = \sum_{\mathrm{p:even}} h^{p,q}$.

Let $\mathfrak{g}_A = \mathrm{Lie}\, G_A$ $(A = \mathbb{R}, \mathbb{C})$. We then have the decomposition $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p+q=0} \mathfrak{g}^{p,q}$ given by

$$\mathfrak{g}^{p,q} = \left\{ \alpha \in \mathfrak{g}_{\mathbb{C}} \mid \alpha H^{p',q'} \subset H^{p+p',q+q'} \text{ for } p', q' \in \mathbb{Z} \right\}$$

with respect to a Hodge decomposition $H_{\mathbb{C}} = \bigoplus H^{p,q}$.

Example 2.1 (Upper half plane)**.** Let us consider the case where the Hodge numbers $h^{1,0} = h^{0,1} = 1$, 0 otherwise. Then corresponding classifying space D is the upper-half plane $\{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$, and $\check{D} \cong \mathbb{P}^1$. $G_A \cong SL(2, A)$ $(A = \mathbb{Z}, \mathbb{R}, \mathbb{C})$ where the action of $SL(2,\mathbb{C})$ on $\check{D}$ is given by the linear fractional transformation. Here $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(2,\mathbb{R})$ is generated by

$$\mathbf{n}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{n}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We call the triple the sl_2-triple. The sl_2-triple satisfies the following conditions:

$$[\mathbf{n}_+, \mathbf{n}_-] = \mathbf{h}, \quad [\mathbf{n}_\pm, \mathbf{h}] = \pm 2\mathbf{n}_\pm.$$

The Hodge decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to $i \in D$ is given by

$$\mathfrak{g}^{-1,1} = \mathbb{C}(i\mathbf{n}_- - \mathbf{h} + i\mathbf{n}_+), \quad \mathfrak{g}^{0,0} = \mathbb{C}(\mathbf{n}_- - \mathbf{n}_+), \quad \mathfrak{g}^{1,-1} = \overline{\mathfrak{g}^{-1,1}}. \tag{2.1}$$

The isotropy subgroup L of $G_{\mathbb{R}}$ at F_0 is given by

$$\begin{aligned} L &= \{g \in G_{\mathbb{R}} \mid gF_0 = F_0\} \\ &\cong \begin{cases} \prod_{p \le m} U(h^{p,q}) & \text{if } w = 2m+1, \\ \prod_{p<m} U(h^{p,q}) \times SO(h^{m,m}) & \text{if } w = 2m. \end{cases} \end{aligned}$$

They are compact subgroups of $G_{\mathbb{R}}$ but not maximal compact unless D is Hermitian symmetric. We define

$$H^{\mathrm{even}} = \bigoplus_{p:\mathrm{even}} H_0^{p,q}, \quad H^{\mathrm{odd}} = \bigoplus_{p:\mathrm{odd}} H_0^{p,q}$$

where $H_{\mathbb{C}} = \bigoplus H_0^{p,q}$ is the Hodge decomposition for F_0. The maximal compact subgroup K cantaining L is given by

$$\begin{aligned} K &= \{g \in G_{\mathbb{R}} \mid gH^{\mathrm{even}} = H^{\mathrm{even}}\} \\ &\cong \begin{cases} U(h) & \text{if } w \text{ is odd,} \\ SO(h_{\mathrm{odd}}) \times SO(h_{\mathrm{even}}) & \text{if } w \text{ is even.} \end{cases} \end{aligned}$$

By the connectivity of $G_{\mathbb{R}}$, D is connected if w is odd, D has two connected component if w is even and $h_{\mathrm{even}}, h_{\mathrm{odd}} > 0$. Here D is Hermitian symmetric if and only if the isotropy subgroup is a maximally compact subgroup, i.e., one of the following is satisfied:

(1) $w = 2m+1$, $h^{p,q} = 0$ unless $p = m+1, m$;
(2) $w = 2m$, $h^{p,q} = 1$ for $p = m+1, m-1$, $h^{m,m}$ is arbitary, $h^{p,q} = 0$ otherwise;
(3) $w = 2m$, $h^{p,q} = 1$ for $p = m+a, m+a-1, m-a, m-a+1$ for some $a \ge 2$, $h^{p,q} = 0$ otherwise.

In the case (1), D is a Hermitian symmetric domain of type III. In the case (2) or (3), an irreducible component of D is a Hermitian symmetric domain of type IV. We call the cases (1)–(3) the classical situation.

Example 2.2 (The weight 1 case)**.** We give an example of period domains of weight 1 and $h^{1,0} = h^{0,1} = n$, 0 otherwise. This case is corresponding to the case (1) above. Here $G_A = Sp(n, A)$ $(A = \mathbb{Z}, \mathbb{R}, \mathbb{C})$ and

$$\begin{aligned} D &= \left\{ \ W : \langle \ , \ \rangle\text{-isotropic } n\text{-planes} \ \middle| \ W \gg 0 \text{ for } i\langle \bullet, \bar{\bullet} \rangle \ \right\} \\ &\cong \{ Z \in \mathbb{C}^{n\times n} \mid I - ZZ^* \gg 0 \} \\ &\cong Sp(n, \mathbb{R})/U(n). \end{aligned} \tag{2.2}$$

See [N] for detail.

2.2. **Cycle spaces of period domains.** Let D_0 be a irreducible component including F_0 of a period domain D. Then $D_0 = G_{\mathbb{R},0}F_0$ where $G_{\mathbb{R},0}$ is identity component of $G_{\mathbb{R}}$. Let K_0 be the maximally compact subgroup of $G_{\mathbb{R},0}$ containing the isotropy subgroup L_0 at F_0. Applying [FHW, Theorem 4.4.3] to D_0, we have the following theorem:

Theorem 2.3. *If D_0 is not Hermitian symmetric domain (i.e., $L_0 \neq K_0$) any holomorphic function on D_0 is constant.*

Proof. We have the non-trivial projection $D_0 \cong G_{\mathbb{R},0}/L_0 \to G_{\mathbb{R},0}/K_0$, which is not holomrphic. In fact $G_{\mathbb{R},0}/K_0$ does not have complex structure if the weight is even and D is not Hermitian symmetric, and if the weight is odd the projection is given by $F \mapsto H^{\mathrm{even}}$ or H^{odd}, which is not holomorphic. Then *the bounded symmetric domain $D(G_{\mathbb{R},0}, F_0)$ subordinate to D_0* ([FHW, Definition 4.4.1]) is a point. Moreover, by [FHW, Theorem 4.4.3], any holomorphic function $\tilde{f}$ on D_0 is written as $\tilde{f} = f \circ \pi$ where $\pi : D_0 \to D(G_{\mathbb{R},0}, F_0)$ and $f : D(G_{\mathbb{R},0}, F_0) \to \mathbb{C}$ is holomorphic. □

We call K_0-orbit $C_0 = K_0 \cdot F_0$ the base cycle of F_0. By [FHW, Theorem 4.3.1], $C_0 = K_{0,\mathbb{C}} \cdot F_0$ and it is a compact submanifold of D.

Proposition 2.4 ([FHW] Lemma 5.1.3)**.** *Let $J = \{g \in G_{\mathbb{C}} \mid gC_0 = C_0\}$. Then J is a closed complex subgroup of $G_{\mathbb{C}}$. The quotient manifold $\mathcal{M}_{\check{D}} = \{gC_0 \mid g \in G_{\mathbb{C}}\} \cong G_{\mathbb{C}}/J$ has a natural structure of $G_{\mathbb{C}}$-homogeneous complex manifold, and the subset $\{gC_0 \mid g \in G_{\mathbb{C}}$ and $gC_0 \subset D\}$ is open in $\mathcal{M}_{\check{D}}$.*

The topological component of C_0 in $\{gC_0 \mid g \in G_{\mathbb{C}}$ and $gC_0 \subset D_0\}$ is called the cycle space of D_0. We denote the cycle space of D_0 by $\mathcal{M}_{D_0}$.

2.3. **Cycle spaces for odd-weight cases.** We describe cycle spaces explicitly in an odd-weight case following [FHW, 5.5B]. In this case $D = D_0$. For a base point $F_0 \in D$ we define

$$f^p_{\mathrm{even}} = \sum_{\substack{r \geq p, \\ r:\ \mathrm{even}}} h^{r,s}, \quad f^p_{\mathrm{odd}} = \sum_{\substack{r \geq p, \\ r:\ \mathrm{odd}}} h^{r,s}.$$

Here the base cycle C_0 is written as

$$C_0 = \{F \in \check{D} \mid \dim (F^p \cap H^{\mathrm{even}}) = f^p_{\mathrm{even}}, \dim (F^p \cap H^{\mathrm{odd}}) = f^p_{\mathrm{odd}}\}.$$

Let V and W be $\langle \ , \ \rangle$-isotropic subspaces, and let

$$C_{V,W} = \{F \in \check{D} | \dim (F^p \cap V) = f^p_{\mathrm{even}}, \dim (F^p \cap W) = f^p_{\mathrm{odd}}\}.$$

Here $C_0 = C_{H^{\mathrm{even}}, H^{\mathrm{odd}}}$ and $gC_{V,W} = C_{gV,gW}$ for $g \in G_{\mathbb{C}}$.

Proposition 2.5.

$$\mathcal{M}_D = \{C_{V,W} \mid V \ll 0 \text{ and } W \gg 0 \text{ for } i^w\langle \bullet, \bar{\bullet}\rangle\} \cong \mathcal{B} \times \bar{\mathcal{B}}$$

where w is the weight.

Proof. By [FHW, Lemma 5.4.1],

$$G_{\mathbb{R}}H^{\mathrm{even}} \times G_{\mathbb{R}}H^{\mathrm{odd}} \subset G_{\mathbb{C}}(H^{\mathrm{even}}, H^{\mathrm{odd}}) \subset G_{\mathbb{C}}H^{\mathrm{even}} \times G_{\mathbb{C}}H^{\mathrm{odd}}.$$

Here $G_{\mathbb{R}}H^{\mathrm{odd}} \cong G_{\mathbb{R}}/K$, which has a realization $G_{\mathbb{R}}H^{\mathrm{odd}} \cong \mathcal{B}$ as a bounded domain $\mathcal{B}$ (Example 2.2). Moreover $G_{\mathbb{R}}H^{\mathrm{even}} \cong \bar{\mathcal{B}}$. Now $G_{\mathbb{C}}(H^{\mathrm{even}}, H^{\mathrm{odd}}) \cong G_{\mathbb{C}}/K_{\mathbb{C}}$, and then $G_{\mathbb{C}}/K_{\mathbb{C}} \supset \mathcal{B} \times \bar{\mathcal{B}}$.

Since $\mathcal{M}_{\check{D}} \cong G_{\mathbb{C}}/J$ and $K_{\mathbb{C}} \subset J$, we have a projection

$$\pi : G_{\mathbb{C}}/K_{\mathbb{C}} \to \mathcal{M}_{\check{D}}; \quad g \pmod{K_{\mathbb{C}}} \mapsto gC_0.$$

By [FHW, Proposition 5.4.3], π is injective on $\mathcal{B}\times\bar{\mathcal{B}}$. Moreover, by [FHW, Theorem 5.5.1], $\pi(\mathcal{B} \times \bar{\mathcal{B}}) = \mathcal{M}_D \subset \mathcal{M}_{\check{D}}$. □

3. Moduli spaces of polarized log Hodge structures

In this section, we review the construction of moduli spaces of log Hodge structures and state the fundamental properties following [KU] in §3.1. We show the main result in §3.2

3.1. **Construction and fundamental properties.** We call $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ a nilpotent cone if it satisfies the following conditions:

(1) σ is a closed cone generated by finite elements of $\mathfrak{g}_{\mathbb{Q}}$;
(2) $N \in \sigma$ is a nilpotent as an endmorphism of $H_{\mathbb{R}}$;
(3) $NN' = N'N$ for any $N, N' \in \sigma$.

For $A = \mathbb{R}, \mathbb{C}$, we denote by σ_A the A-linear span of σ in $\mathfrak{g}_A$.

Definition 3.1. Let $\sigma = \sum_{j=1}^{n} \mathbb{R}_{\geq 0} N_j$ be a nilpotent cone and $F \in \check{D}$. $\exp(\sigma_{\mathbb{C}})F \subset \check{D}$ is a σ-nilpotent orbit if it satisfies the following conditions:

(1) $\exp(\sum_j iy_j N_j)F \in D$ for all $y_j \gg 0$.
(2) $NF^p \subset F^{p-1}$ for all $p \in \mathbb{Z}$ and for all $N \in \sigma$.

We define the set of nilpotent orbits

$$D_\sigma := \{(\tau, Z) \mid \tau\text{: face of } \sigma,\ Z \text{ is a } \tau\text{-nilpotent orbit}\}.$$

For a nilpotent cone σ, we have the abelian group and the monoid

$$\Gamma(\sigma)^{\mathrm{gp}} = \exp(\sigma_{\mathbb{R}}) \cap G_{\mathbb{Z}}, \quad \Gamma(\sigma) = \exp(\sigma) \cap G_{\mathbb{Z}}.$$

We define a geometric structure on $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$. First we review some basic facts about toric varieties. The monoid $\Gamma(\sigma)$ define the toric varieties

$$\mathrm{toric}_\sigma := \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^\vee])_{\mathrm{an}} \cong \mathrm{Hom}(\Gamma(\sigma)^\vee, \mathbb{C}),$$
$$\mathrm{torus}_\sigma := \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^{\vee\mathrm{gp}}])_{\mathrm{an}} \cong \mathrm{Hom}(\Gamma(\sigma)^{\vee\mathrm{gp}}, \mathbb{G}_m) \cong \mathbb{G}_m \otimes \Gamma(\sigma)^{\mathrm{gp}},$$

where $\mathbb{C}$ in the right hand side of the first line is regarded as a semigroup via multiplication and above homomorphisms are of semigroups. As in [F, §2.1], we choose for a face τ of σ the distinguished point

$$x_\tau : \Gamma(\sigma)^\vee \to \mathbb{C}; \quad u \mapsto \begin{cases} 1 & \text{if } u \in \Gamma(\tau)^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Then toric_σ can be decomposed by torus orbits as

$$\mathrm{toric}_\sigma = \bigsqcup_{\tau\text{: face of } \sigma} (\mathrm{torus}_\sigma \cdot x_\tau).$$

For $q \in \mathrm{toric}_\sigma$, there exists the face $\sigma(q)$ of σ such that $q \in \mathrm{torus}_\sigma \cdot x_{\sigma(q)}$. By a surjective homomorphism

$$\mathbf{e} : \sigma_{\mathbb{C}} \to \mathrm{torus}_\sigma \cong \mathbb{G}_m \otimes \Gamma(\sigma)^{\mathrm{gp}};\ w \log(\gamma) \mapsto \exp(2\pi\sqrt{-1}w) \otimes \gamma,$$

q can be written as $q = \mathbf{e}(z) \cdot x_{\sigma(q)}$. Here $\ker(\mathbf{e}) = \log(\Gamma(\sigma)^{\mathrm{gp}})$ and z is determined uniquely modulo $\log(\Gamma(\sigma)^{\mathrm{gp}}) + \sigma(q)_{\mathbb{C}}$.

We define the analytic space $\check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D}$ and the subset

$$E_\sigma := \left\{ (q, F) \in \check{E}_\sigma \;\middle|\; \begin{array}{r} \exp(\sigma(q)_{\mathbb{C}}) \exp(z)F \text{ is } \sigma(q)\text{-nilpotent orbit} \\ \text{where } q = \mathbf{e}(z) \cdot x_{\sigma(q)}. \end{array} \right\}.$$

Here we endow E_σ with the *strong topology* ([KU, §3.1]) in $\check{E}_\sigma$. We then define the canonical map

$$\begin{aligned} \pi : &E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma, \\ &(q, F) \mapsto (\sigma(q), \exp(\sigma(q)_{\mathbb{C}}) \exp(z)F) \mod \Gamma(\sigma)^{\mathrm{gp}}. \end{aligned}$$

We endow $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ with the strongest topology for which the maps π are continuous. [KU] gives the geometric properties of E_σ, $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ and $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ by using the language *log manifolds* ([KU, §3.5]):

Theorem 3.2 ([KU, Theorem A]). *(1) E_σ and $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ are logarithmic manifolds.*

(2) We have the $\sigma_{\mathbb{C}}$-action on E_σ over $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ by

$$a \cdot (q, F) := (\mathbf{e}(a)q, \exp(-a)F) \quad (a \in \sigma_{\mathbb{C}},\ (q, F) \in E_\sigma),$$

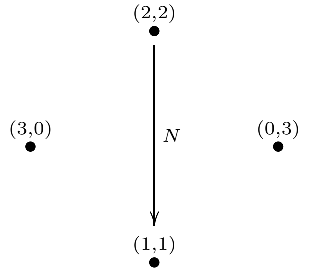
and $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is a $\sigma_{\mathbb{C}}$-torsor in the category of logarithmic manifold.

Moreover [KU] defines *polarized log Hodge structures* ([KU, §2.4]), and they show $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is a fine moduli space of polarized log Hodge structures ([KU, Theorem B]).

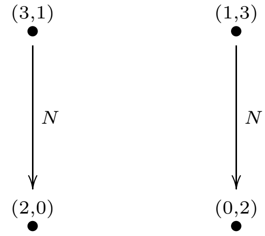
Log manifolds are roughly analytic spaces with *slits*. In the classical situation, $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is just a toroidal partial compactification and the boundary is of codimension 1 (see [N]). However, the codimension may be higher than 1 in the non-classical situation.

Example 3.3 (The (1,1,1,1)-case). Nilpotent orbits in the case where the weight is 3 and the Hodge number is $h^{3,0} = h^{0,3} = 1$ and $h^{1,2} = h^{2,1} = 1$, 0 otherwise (we call it the (1,1,1,1)-case) are classified by [KU, §12.3] or [GGK]. In this case $D \cong Sp(2, \mathbb{R})/(U(1) \times U(1))$ and $\dim D = 4$. Here D is not Hermitian symmetric space. All possible nilpotent cones are of rank 1. For a nilpotent orbit $(\mathbb{R}_{\geq 0}N, \exp(\mathbb{C}N)F)$, we have the limiting mixed Hodge structure $(W(N), F)$ by [S]. Here $(W(N), F)$ is one of the following types:

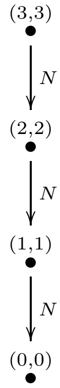
Type-I: $N^2 = 0,\ \dim(\mathrm{Im}\,N) = 1.$



Type-II: $N^2 = 0,\ \dim(\mathrm{Im}\,N) = 2.$



Type-III: $N^3 \neq 0, N^4 = 0.$



Dimensions of boundaries

	$\dim(D_\sigma - D)$
Type-I	2
Type-II	1
Type-III	1

Geometrically type-I or type-III degeneration occur in the quintic-mirror family, and type-II degeneration occur in the Borcea-Voisin mirror family (see [GGK, Part III. A], [U]).

3.2. **Whether the torsors are trivial.** By Theorem 3.2, we have the torsor $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ for a period domain D and a nilpotent cone σ. In [H], we showed triviality of torsors in the classical situation. We show non-triviality of the torsors in the non-classical situation by using the fact that any holomorphic functions on D is constant in the non-classical case (Theorem 2.3).

Theorem 3.4. *Let D be a period domain (for pure Hodge structures) and let (σ, Z) be nilpotent orbit. Then $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is trivial if and only if D is Hermitian symmetric or $\sigma = \{0\}$.*

Proof. [H, Theorem 5.6] shows a triviality of the torsor for a Hermitian symmetric space D. If $\sigma = \{0\}$, the torsor is just an identity map $D \to D$, therefore the torsor is trivial. Then it is suffice to show that the torsor is non-trivial if D is not Hermitian symmetric.

We assume that $\pi : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is trivial for a non-Hermitian symmetric space D and for a nilpotent cone $\sigma \neq \{0\}$. Now

$$\pi^{-1}(\Gamma(\sigma)^{\mathrm{gp}}\backslash D) = E_\sigma \cap (\mathrm{torus}_\sigma \times \check{D})$$

by the definition of E_σ, and this is a complex analytic space since $\mathrm{torus}_\sigma \times \check{D}$ has trivial log structure. Then the restriction of the torsor to $\pi^{-1}(\Gamma(\sigma)^{\mathrm{gp}}\backslash D)$ is a torsor in the category of complex analytic spaces, and we have a section $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to E_\sigma$

and a holomorphic map $\Phi : D \to (\mathbb{C}^*)^l$ such that

$$
\begin{array}{ccccccc}
 & & \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma & \longrightarrow & E_\sigma & & \\
 & & \cup & & \cup & & \\
\Phi : D & \xrightarrow{\text{quot.}} & \Gamma(\sigma)^{\mathrm{gp}}\backslash D & \longrightarrow & E_\sigma \cap (\mathrm{torus}_\sigma \times \check{D}) & \xrightarrow{\text{proj.}} & \mathrm{torus}_\sigma \cong (\mathbb{C}^*)^l
\end{array}
\tag{3.1}
$$

where $l = \operatorname{rank} \Gamma(\sigma)^{\mathrm{gp}}$.

For a nilpotent N in the relative interior of σ, we have

$$\lim_{y\to\infty} \exp(iyN)F = (\sigma, \exp(\sigma_{\mathbb{C}})F) \tag{3.2}$$

through $D \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D \hookrightarrow \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ by [KU, Proposition 3.4.4]. Then $\Phi(\exp(iyN)F)$ has to be converged in $0 \in \mathrm{toric}_\sigma$ as $y \to \infty$. This contradicts to Theorem 2.3. $\square$

4. Remarks on [H]

We showed a non-triviality of the torsor in [H, Proposition 5.8] by the different way from Theorem 3.4. We formulate it by using SL(2)-orbit theorem and cycle spaces. At first we show a property of some cycle spaces in Lemma 4.1. Using it we show Proposition 4.3 which is a generalization of [H, Proposition 5.8]. Moreover we observe a property of cycle spaces connecting to Lemma 4.1 in the (1,1,1,1)-case in §4.3. In this section we assume D is not Hermitian symmetric.

4.1. **SL(2)-orbits and cycle spaces.** Let $(\mathbb{R}_{\geq 0}N, \exp(\mathbb{C}N)F)$ be a nilpotent orbit. By [S] there exists the monodromy weight filtration $W(N)$ and $(W(N), F)$ is a mixed Hodge structure. By [CKS, Proposition 2.20] there exists the $\mathbb{R}$-split mixed Hodge structure $(W(N), \hat{F})$ associated to it. We then have the Deligne decomposition $H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$ for $(W(N), \hat{F})$ where

$$\hat{F}^p = \bigoplus_{r\leq p} I^{r,s}, \quad W(N)_k = \bigoplus_{r+s=k} I^{r,s}, \quad \overline{I^{p,q}} = I^{q,p}.$$

By SL(2)-orbit theorem ([S, Theorem 5.13], [CKS, §3]), there exists the Lie group homomorphism $\rho : SL(2,\mathbb{C}) \to G_{\mathbb{C}}$ defined over $\mathbb{R}$ and the holomorphic map $\phi : \mathbb{P}^1 \to \check{D}$ satisfying the following conditions:

- (S1) $\rho(g)\phi(z) = \phi(gz)$;
- (S2) $\phi(0) = \hat{F}$;
- (S3) $\rho_*(\mathbf{n}_-) = N$;
- (S4) $Hv = (p+q-w)v$ for $v \in I^{p,q}$ where $\rho_*(\mathbf{h}) = H$;
- (S5) $\rho_* : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$ is a $(0,0)$-morphism of Hodge structure where $\mathfrak{g}_{\mathbb{R}}$ (resp. $\mathfrak{sl}(2,\mathbb{R})$) has a Hodge structure of weight 0 relative to $\phi(i)$ (resp. i),

where $\{\mathbf{n}_-, \mathbf{h}, \mathbf{n}_+,\}$ is the sl_2-triple (Example 2.1).

Let $F_0 = \phi(i)$ be a base point of D. We write

$$\rho_*(\mathbf{n}_+) = N^+, \quad X = \frac{1}{2}(iN - H + iN^+).$$

Then $X \in \mathfrak{g}_0^{-1,1}$ by (S5) and (2.1) where $\mathfrak{g}_0^{-1,1}$ is the $(-1,1)$-component of Hodge decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to F_0. By (S1) we have

$$\exp(zX)\phi(i) = \phi\left(\frac{1+z}{1-z}i\right),$$

therefore

$$\exp\left(\frac{y}{2+y}X\right)\phi(i) = \phi((1+y)i) = \exp(iyN)\phi(i). \tag{4.1}$$

Lemma 4.1. *Let C_0 be the base cycle of F_0. If $\dim(\mathrm{Im}\, N) = 1$, then both (A) and (B) hold:*

(A) There exists $F_{\mathrm{fix}} \in C_0$ such that $\exp(X)F_{\mathrm{fix}} = F_{\mathrm{fix}}$;
(B) $\exp(zX)C_0 \subset D$ (i.e., $\exp(zX)C_0 \in \mathcal{M}_D$) for $|z| < 1$.

Proof. At first we write X explicitly. Considering the type of limiting Hodge structure $(W(N), F)$, the case where $\dim(\mathrm{Im}\, N) = 1$ is possibly if the weight is $2m-1$ and $\dim(\mathrm{Gr}^W_{2m}) = \dim(\mathrm{Gr}^W_{2m-2}) = 1$. We then have a $\mathbb{R}$-element e in the (m,m)-component $I^{m,m}$ of the Deligne decomposition of $(W(N), \hat{F})$. Here X is given by

$$e \mapsto \frac{1}{2}(-e + iNe), \quad Ne \mapsto \frac{1}{2}(ie + Ne), \quad I^{p,q} \to 0 \quad \text{for } p+q = 2m-1.$$

We write $u = \exp(iN)e$. Since $e \in \hat{F}^m$, $u \in \exp(iN)\hat{F}^m = F_0^m$. Moreover, since $Ne \in \hat{F}^{m-1}$,

$$Ne = \exp(iN)Ne \in \exp(iN)\hat{F}^{m-1} = F_0^{m-1}.$$

Then

$$\bar{u} = e - iNe = u - 2iNe \in F_0^{m-1}.$$

Hence u is in the $(m, m-1)$-component $H_0^{m,m-1}$ of the Hodge decomposition for F_0. Here

$$Xu = -e + iNe = -\bar{u}, \quad Xw = 0 \text{ if } \langle w, \bar{u}\rangle = 0. \tag{4.2}$$

We show (A). We denote by $\|\bullet\|$ the norm induced by the positive definite Hermitian form $\langle C_{F_0}\bullet, \bar{\bullet}\rangle$ where C_{F_0} is Weil operator for F_0. Scaling u, we may assume $\|u\| = 1$. We take $v \in H_0^{m-2,m+1}$ such that $\|v\| = 1$. We define $g \in \mathrm{Aut}(H_{\mathbb{C}})$ by

$$gu = v, \quad gv = u, \quad g\bar{v} = \bar{u}, \quad g\bar{u} = \bar{v},$$

and $gw = w$ if w is vertical to $u, v, \bar{u}$ and $\bar{v}$ for $\langle\ ,\ \rangle$. Then

$$gu = v = \overline{g\bar{u}} = \bar{g}u, \quad gv = u = \overline{g\bar{v}} = \bar{g}v.$$

Therefore g is defined over $\mathbb{R}$ and preserves the polarization $\langle\ ,\ \rangle$ i.e. $g \in G_{\mathbb{R}}$. Moreover $g \in K$ since g preserves H^{even}.
Claim. $gF_0 \in C_0$ is a fixed point for $\exp(X)$.

Proof. Now $u = gv \in gH_0^{m-2,m+1}$. By (4.2) it is suffice to show that $Xu \in gF_0^{m-2}$. In fact

$$Xu = -\bar{u} = -g\bar{v} \in gH_0^{m+1,m-2}.$$

□

Next, we show (B). We take a unitary basis $\{u_1, \ldots, u_l\}$ of $H_0^{m,m-1}$. We may assume $u_1 = u$. Then $\exp(X)u_j = u_j$ if $j \neq 1$ and $\exp(zX)u_1 = u_1 - z\bar{u}_1$. Here

$$i\langle \exp(zX)u_1, \overline{\exp(zX)u_1}\rangle = \|u_1\|^2 - |z|^2\|u_1\|^2 = 1 - |z|^2.$$

By Proposition 2.5, $\exp(zX)C_0 \subset D$ if and only if $|z| < 1$. □

Remark 4.2. Above (4.1), (A) and (B) are corresponding to the conditions (5.4), (5.6) and (5.5) of [H, §5] respectively.

4.2. **Non-triviality on some open sets.** Let $(\mathbb{R}_{\geq 0}N, \exp(\mathbb{C}N)F)$ be a nilpotent orbit. Let (ρ, ϕ) be the SL(2)-orbit associated to (N, F). We take a base point $F_0 = \phi(i)$, $X \in \mathfrak{g}_0^{-1,1}$ and the base cycle C_0 of $F_0 = \phi(i)$. We define

$$\mathcal{M}(\varepsilon) = \{\exp(\alpha X)C_0 \mid 1 - \varepsilon < \alpha < 1\} \subset \mathcal{M}_{\check{D}}.$$

for $0 < \varepsilon$. If $\dim(\operatorname{Im} N) = 1$, by Lemma 4.1 (B)

$$\exp(\alpha X)C_0 \in \mathcal{M}_D \text{ for } -1 < \alpha < 1, \quad \exp(X)C_0 \notin \mathcal{M}_D.$$

Then $\mathcal{M}(\varepsilon)$ is a nearby set of the boundary point $\exp(X)C_0 \in \overline{\mathcal{M}}_D$.

Proposition 4.3. *Let U be an open set including the boundary point $(\sigma, \exp(\sigma_{\mathbb{C}})\hat{F})$ in $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ where $\sigma = \mathbb{R}_{\geq 0}N$ with $\dim(\operatorname{Im} N) = 1$. If there exists $0 < \varepsilon < 1$ such that $q(C) \subset U$ for $C \in \mathcal{M}(\varepsilon)$ and for the quotient map $q : D \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D$, then no section over the open set U exists.*

Proof. We assume there exists a local trivialization over U. Similar to the proof of Theorem 3.4, we have a section $U \to E_\sigma$ and a holomorphic map $\Phi : q^{-1}(U) \to \mathbb{C}^*$ such that

$$\begin{array}{ccccccc} & & U & \longrightarrow & E_\sigma & & \\ & & \cup & & \cup & & \\ \Phi : q^{-1}(U) & \longrightarrow & U \cap (\Gamma(\sigma)^{\mathrm{gp}}\backslash D) & \longrightarrow & E_\sigma \cap (\mathbb{C}^* \times \check{D}) & \longrightarrow & \mathbb{C}^*. \end{array}$$

By (4.1) and the assumption,

$$q\left(\exp\left(\frac{y}{2+y}X\right)F_0\right) = q(\exp(iyN)F_0) \subset U \quad \text{for } \frac{2(1-\varepsilon)}{\varepsilon} \leq y.$$

By (3.2), $\Phi(\exp(iyN)F_0)$ has to be converged to 0 as $y \to \infty$.

Now Φ is constant on the compact complex submanifold $C \in \mathcal{M}(\varepsilon)$. By Lemma 4.1, we then have

$$\begin{aligned} \Phi(\exp(iyN)F_0) &= \Phi\left(\exp\left(\frac{y}{2+y}X\right)F_0\right) = \Phi\left(\exp\left(\frac{y}{2+y}X\right)F_{\mathrm{fix}}\right) \\ &= \Phi\left(\exp\left(\frac{y'}{2+y'}X\right)F_{\mathrm{fix}}\right) = \Phi(\exp(iy'N)F_0) \end{aligned}$$

for $y, y' > 2(1-\varepsilon)/\varepsilon$. This contradicts to the convergence of $\Phi(\exp(iyN)F_0)$. $\square$

Remark 4.4. Above X, F_{fix} and F_0 are corresponding to the notations N', F_∞ and F_0 in [H, §5] respectively.

4.3. **The (1,1,1,1)-case.** A property of cycle spaces of Lemma 4.1 induces Proposition 4.3. In the (1,1,1,1)-case, it is applied to the type-I nilpotent orbits. We show that (A) or (B) of Lemma 4.1 does not hold in other types.

Let $(\mathbb{R}_{\geq 0}N, \exp(\mathbb{C}N)F)$ be a nilpotent orbit and let (ρ, ϕ) be the SL(2)-orbit associated (N, F). We can choose a unitary basis

$$u_3 \in H_0^{3,0}, \quad u_2 \in H_0^{2,1}, \quad \bar{u}_2 \in H_0^{1,2}, \quad \bar{u}_3 \in H_0^{0,3}$$

for the Hodge decomposition for $F_0 = \phi(i)$. Here the base cycle of F_0 is $C_0 \cong U(2)/(U(1) \times U(1)) \cong \mathbb{P}^1$. The isomorphism $\mathbb{P}^1 \xrightarrow{\sim} C_0 \subset D$ is given by

$$\begin{aligned} &F_z^3 = \operatorname{span}_{\mathbb{C}}\{z\bar{u}_2 + u_3\}, \quad F_z^2 = \operatorname{span}_{\mathbb{C}}\{z\bar{u}_2 + u_3, u_2 - z\bar{u}_3\}, \\ &F_\infty^3 = \operatorname{span}_{\mathbb{C}}\{\bar{u}_2\}, \quad F_\infty^2 = \operatorname{span}_{\mathbb{C}}\{\bar{u}_2, \bar{u}_3\}. \end{aligned} \tag{4.3}$$

The properties (A) and (B) of Lemma 4.1 depend on $X \in \mathfrak{g}_0^{-1,1}$. For a type-I nilpotent, X is given by

$$\overset{(3,0)}{\bullet} \qquad \overset{(2,1)}{\bullet} \xrightarrow{X} \overset{(1,2)}{\bullet} \qquad \overset{(0,3)}{\bullet} \qquad (u_2 \mapsto -\bar{u}_2 \mapsto 0).$$

We determine the type of X in the case for type-II and for type-III, and consider whether (A) or (B) holds or not.

4.3.1. *type-II.*

Proposition 4.5. *If N is of type-II, then (B) holds, however (A) does not hold.*

Proof. Let v be a non-zero element in $I^{3,1}$ of the Deligne decomposition of $(W(N), \hat{F})$. Then

$$Nv \in I^{2,0}, \quad \bar{v} \in I^{1,3}, \quad N\bar{v} \in I^{0,2}.$$

We write $u_3 = \exp{(iN)}v$. Since $v \in \hat{F}^3$, $u_3 \in F_0^3 = H_0^{3,0}$. Here the sl_2-triple is given by

$$\begin{aligned} &N^+Nv = v, \quad N^+N\bar{v} = \bar{v}, \quad N^+v = N^+\bar{v} = 0, \\ &Hv = v, \quad H\bar{v} = \bar{v}, \quad HNv = -Nv, \quad HN\bar{v} = -N\bar{v}. \end{aligned}$$

Then we have

$$H_0^{2,1} \ni Xu_3 = -v + iNv = -\exp{(-iN)}v.$$

We write $u_2 = Xu_3$. Then $Xu_2 = 0$. Moreover $\bar{u}_2 \in H_0^{1,2}$, and

$$X\bar{u}_2 = \bar{v} - iN\bar{v} = \bar{u}_3.$$

Summarizing these, $X \in \mathfrak{g}_0^{-1,1}$ is given by

$$\overset{(3,0)}{\bullet} \xrightarrow{X} \overset{(2,1)}{\bullet} \qquad \overset{(1,2)}{\bullet} \xrightarrow{X} \overset{(0,3)}{\bullet} \qquad (u_3 \mapsto u_2 \mapsto 0,\ \bar{u}_2 \mapsto \bar{u}_3 \mapsto 0).$$

Since $X(z\bar{u}_2 + u_3) = z\bar{u}_3 + u_2$, $XF_z^3 \not\subset F_z^3$ for $z \in \mathbb{P}^1$ in (4.3). Then there is no fixed point for $\exp{(X)}$ in C_0.

Next we show (B) holds. Scaling v, we may assume $\|u_3\| = 1$.
Claim. $\|u_2\| = 1$.

Proof. Let $a = \langle v, \bar{v}\rangle$, $b = \langle Nv, \bar{v}\rangle$, $c = \langle v, N\bar{v}\rangle$ and $d = \langle Nv, N\bar{v}\rangle$. Then by the orthogonality

$$\begin{aligned} &\langle u_3, \bar{u}_3\rangle = a + ib - ic + d = i, \quad \langle u_3, \bar{u}_2\rangle = -a - ib - ic + d = 0, \\ &\langle u_2, \bar{u}_3\rangle = -a + ib + ic + d = 0. \end{aligned}$$

Since $v \in \hat{F}^3$ and $\bar{v} \in \hat{F}^1$, $a = 0$. Therefore the simultaneous equation induces $d = 0$, $b - c = 1$ and $\langle u_2, \bar{u}_2\rangle = a - ib + ic + d = -i$. □

Here $\{u_3, u_2, \bar{u}_3, \bar{u}_2\}$ is a unitary basis. Since

$$\begin{aligned} &-i\langle \exp{(zX)}u_3, \overline{\exp{(zX)}u_3}\rangle = \|u_3\|^2 - |z|^2\|u_2\|^2 = 1 - |z|^2, \\ &-i\langle \exp{(zX)}\bar{u}_2, \overline{\exp{(zX)}\bar{u}_2}\rangle = \|u_2\|^2 - |z|^2\|u_3\|^2 = 1 - |z|^2, \end{aligned}$$

$\exp{(zX)}C_0 \subset D$ if and only if $|z| < 1$ by Proposition 2.5. □

4.3.2. *type-III.* We give an example of type-III which satisfies neither (A) nor (B). All nilpotent orbits of type-III are described in [GGK] explicitly. We consider the case where $a, b = 1$ and $e, f, \pi = 0$ in the notation of [GGK, (I.C.2), (I.C.10)]. Let $H_{\mathbb{Z}} = \sum_{j=0}^{3} \mathbb{Z} e_j$. We write

$$e_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where a bilinear form is given by

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \hat{F}^p = \{e_3, \dots, e_p\} (3 \geq p \geq 0).$$

Then N and $\hat{F}$ give a nilpotent orbit of type-III, where the limit mixed Hodge structure $(W(N), \hat{F})$ is $\mathbb{R}$-split.

The sl_2-triple of the SL(2)-orbit associated to this nilpotent orbit is given by

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad N^+ = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$X = \frac{1}{2} \begin{pmatrix} -3 & 3i & 0 & 0 \\ i & -1 & 4i & 0 \\ 0 & i & 1 & -3i \\ 0 & 0 & -i & 3 \end{pmatrix}.$$

Proposition 4.6. *For the above example, both (A) and (B) do not hold.*

Proof. Let

$$u_3 = \frac{\sqrt{3}}{2} \exp(iN) e_3 = \frac{\sqrt{3}}{12} \begin{pmatrix} 6 \\ 6i \\ -3 \\ i \end{pmatrix}.$$

Then $\|u_3\| = 1$. Now

$$X u_3 = \frac{\sqrt{3}}{4} \begin{pmatrix} -6 \\ -2i \\ -1 \\ i \end{pmatrix}, \quad X^2 u_3 = \frac{\sqrt{3}}{2} \begin{pmatrix} 6 \\ -2i \\ 1 \\ i \end{pmatrix} = -2\overline{X u_3},$$

$$X^3 u_3 = \frac{\sqrt{3}}{2} \begin{pmatrix} -6 \\ 6i \\ 3 \\ i \end{pmatrix} = -6 \bar{u}_3.$$

Here $\|Xu_3\| = 3$. Letting $u_2 = \frac{1}{\sqrt{3}}Xu_3$, we then have a unitary basis $\{u_3, u_2, \bar{u}_3, \bar{u}_2\}$. X gives the map

$$\underset{\bullet}{(3,0)} \xrightarrow{X} \underset{\bullet}{(2,1)} \xrightarrow{X} \underset{\bullet}{(1,2)} \xrightarrow{X} \underset{\bullet}{(0,3)}$$

$$(u_3 \mapsto \sqrt{3}u_2 \mapsto -2\sqrt{3}\,\overline{u_2} \mapsto -6\,\overline{u_3} \mapsto 0).$$

Then $XF_z^3 \not\subset F_z^3$ for $z \in \mathbb{P}^1$ in (4.3), and so there is no fixed point in C_0. Moreover, for $\bar{u}_2 \in F_\infty^3$

$$-i\langle \exp{(zX)}\bar{u}_2, \overline{\exp{(zX)}\bar{u}_2}\rangle = \|u_2\|^2 - 3|z|^2\|u_3\|^2 = 1 - 3|z|^2.$$

Then $\exp{(zX)}C_0 \not\subset D$ for $|z| \geq 1/\sqrt{3}$. $\square$

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Department of Mathematics, National Taiwan University, Taipei 106, Taiwan
E-mail address: t-hayama@math.ntu.edu.tw